Introduction to cell-centered Lagrangian schemes

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1 Introduction

2 Gas dynamics system of equations

3 First-order numerical scheme for the 2D gas dynamics

4 High-order extension in the 2D case

5 Numerical results in 2D
Eulerian formalism (spatial description)
- fixed referential attached to the observer
- fixed observation zone through the fluid flows

Lagrangian formalism (material description)
- moving referential attached to the material
- observation zone moved and deformed as the fluid flows

Lagrangian formalism advantages
- adapted to problems undergoing large deformations
- naturally tracks interfaces in multi-material flows
- avoids the numerical diffusion of the convection terms

Lagrangian formalism drawbacks
- Robustness issue in the case of strong vorticity or shear flows
  \[ \Rightarrow \] ALE method (Arbitrary Lagrangian-Eulerian)
Cell-centered formulation

Staggered formulation

\( \rho \ \varepsilon \ u \)

\( \Omega_c \)

\( \Omega_p \)
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Gas dynamics system of equations

Eulerian description

Definitions

- $\rho$ the fluid density
- $u$ the fluid velocity
- $e$ the fluid specific total energy
- $p$ the fluid pressure
- $\varepsilon = e - \frac{1}{2} u^2$ the fluid specific internal energy

Euler equations

\[
\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \ u) = 0
\]

\[
\frac{\partial \rho \ u}{\partial t} + \nabla_x \cdot (\rho \ u \otimes u + p \ I_d) = 0
\]

\[
\frac{\partial \rho \ e}{\partial t} + \nabla_x \cdot (\rho \ u \ e + p \ u) = 0
\]

Thermodynamical closure

- $p = p(\rho, \varepsilon)$

Equation of state
Moving referential

- $X$ is the position of a point of the fluid in its initial configuration
- $x(X, t)$ is the actual position of this point, moved by the fluid flow

Trajectory equation

\[
\frac{\partial x(X, t)}{\partial t} = u(x(X, t), t)
\]

- $x(X, 0) = X$

Material derivative

- $f(x, t)$ is a smooth fluid variable
\[
\frac{df(x, t)}{dt} = \frac{\partial f(x, t)}{\partial t} + u \cdot \nabla_x f(x, t)
\]
Definitions

- \( \tau = \frac{1}{\rho} \) the specific volume
- \( U = (\tau, u, e)^t \) the solution vector
- \( F(U) = (u, 1(1) \rho, 1(2) \rho, 1(3) \rho, \rho u)^t \) where \( 1(i) = (\delta_{i1}, \delta_{i2}, \delta_{i3})^t \)
- \( a = a(\rho, \varepsilon) \) the sound speed

Updated Lagrangian formulation

- \( \rho \frac{dU}{dt} + \nabla_x \cdot F(U) = 0 \)

Moving configuration

Non-conservative formulation

- \( \rho \frac{dU}{dt} + A_x(U) \frac{\partial U}{\partial x} + A_y(U) \frac{\partial U}{\partial y} + A_z(U) \frac{\partial U}{\partial z} = 0 \)
- \( A_n = A_x n_x + A_y n_y + A_z n_z \) with \( n \) a unit vector
- \( \lambda(U) = \{-\rho a, 0, \rho a\} \) the eigenvalues of \( A_n(U) \)
### Deformation gradient tensor
- \( J = \nabla_x x \)  
- \( |J| = \det J > 0 \)  
- \( \nabla_x . (|J|J^{-t}) = 0 \)

### Jacobian of the fluid flow
- Positive control volume
- Piola compatibility condition

### Mass conservation
- \( \int_{\omega(0)} \rho^0 \, dV = \int_{\omega(t)} \rho \, dv \)
- \( \int_{\omega(t)} \rho \, dv = \int_{\omega(0)} \rho \, |J| \, dV \)
- \( \rho \, |J| = \rho^0 \)

### Total Lagrangian formulation
- \( \rho^0 \frac{dU}{dt} + \nabla_x . (|J|J^{-1}F(U)) = 0 \)
- Fixed configuration
Introduction

Gas dynamics system of equations

First-order numerical scheme for the 2D gas dynamics

High-order extension in the 2D case

Numerical results in 2D
Définitions

- $0 = t^0 < t^1 < \cdots < t^N = T$ : a partition of the time domain $[0, T]$
- $\omega^0 = \bigcup_{c=1}^l \omega_c^0$ : a partition of the initial domain $\omega^0$
- $\omega_c^n$ : the image of $\omega_c^0$ at time $t^n$ through the fluid flow
- $m_c$ : the constant mass of cell $\omega_c$
- $U_c^n = (\tau_c^n, u_c^n, e_c^n)^t$ : the discrete solution

(a) Straight line edges
(b) Conical edges
(c) Polynomial edges

Figure: Generic polygonal cell
Integration

- \( U_{c}^{n+1} = U_{c}^{n} - \frac{\Delta t^{n}}{m_{c}} \int_{\partial \omega_{c}} \overline{F} \cdot n \, ds \)

- Integration of the cell boundary term (analytically, quadrature, ...)

General first-order finite volumes scheme

- \( U_{c}^{n+1} = U_{c}^{n} - \frac{\Delta t^{n}}{m_{c}} \sum_{q \in Q_{c}} \overline{F}_{qc} \cdot l_{qc} n_{qc} \)

- \( \overline{F}_{qc} = (-\overline{u}_{q}, 1(1) \overline{p}_{qc}, 1(2) \overline{p}_{qc}, \overline{p}_{qc} \overline{u}_{q})^{t} \) numerical flux at point \( q \)

- \( x_{q}^{n+1} = x_{q}^{n} + \Delta t^{n} \overline{u}_{q} \)

Definitions

- \( Q_{c} \) the chosen control point set of cell \( \omega_{c} \)

- \( l_{qc} n_{qc} \) some normals to be defined
Remark

- $\overline{F}_{qc}$ is local to the cell $\omega_c$
- Only $\overline{u}_{qc} = \overline{u}_q$ needs to be continuous, to advect the mesh
- Loss of the scheme conservation?

Figure: Points neighboring cell sets

1D numerical fluxes

- $\overline{p}_{qc} = p^n_c - \tilde{z}_{qc} (\overline{u}_q - u^n_c) \cdot n_{qc}$
- $\tilde{z}_{qc} > 0$ local approximation of the acoustic impedance
Conservation

- \( \sum_c m_c \mathbf{U}_c^{n+1} = \sum_c m_c \mathbf{U}_c^n + \text{BC} \)

- For sake of simplicity, we consider BC = 0

- Necessary condition: \( \sum_c \sum_{q \in Q_c} \bar{p}_{qc} l_{qc} n_{qc} = 0 \)

Example of a solver: LCCDG schemes

- Conditions suffisantes

  \( \forall p \in \mathcal{P}(\omega), \sum_{c \in C_p} [\bar{p}_{pc} l_{pc} n_{pc} + \bar{p}_{pc} l_{pc} n_{pc}^+] = 0 \)

  \[ \Rightarrow \bar{u}_p = \left( \sum_{c \in C_p} M_{pc} \right)^{-1} \sum_{c \in C_p} \left( M_{pc} \mathbf{U}_c^n + p_c^n l_{pc} n_{pc} \right) \]

- \( \forall q \in \mathcal{Q}(\omega) \setminus \mathcal{P}(\omega), (\bar{p}_{qL} - \bar{p}_{qR}) l_{qL} n_{qL} = 0 \iff \bar{p}_{qL} = \bar{p}_{qR} \)

  \[ \Rightarrow \bar{u}_q = \left( \frac{\tilde{z}_{qL} \mathbf{U}_L^n + \tilde{z}_{qR} \mathbf{U}_R^n}{\tilde{z}_{qL} + \tilde{z}_{qR}} \right) - \frac{p_R^n - p_L^n}{\tilde{z}_{qL} + \tilde{z}_{qR}} n_{qf_{pp+}} \]
Convex combination

\[
U_{c}^{n+1} = U_{c}^{n} - \frac{\Delta t^{n}}{m_c} \sum_{q \in Q_c} \bar{F}_{qc} \cdot l_{qc} n_{qc} + \frac{\Delta t^{n}}{m_c} F(U_{c}^{n}) \cdot \sum_{q \in Q_c} l_{qc} n_{qc} = 0
\]

\[
U_{c}^{n+1} = (1 - \lambda_{c}) U_{c}^{n} + \sum_{q \in Q_c} \lambda_{qc} \bar{U}_{qc}
\]

Definitions

\[
\lambda_{qc} = \frac{\Delta t^{n}}{m_c} \tilde{z}_{qc} l_{qc} \quad \text{and} \quad \lambda_{c} = \sum_{q \in Q_c} \lambda_{qc}
\]

\[
\bar{U}_{qc} = U_{c}^{n} - \frac{\bar{F}_{qc} - F(U_{c}^{n})}{\tilde{z}_{qc}} \cdot n_{qc}
\]

CFL condition

\[
\Delta t^{n} \leq \frac{m_c}{\sum_{q \in Q_c} \tilde{z}_{qc} l_{qc}} \left( = \frac{|\omega|}{a_{c}^{n}} \right) \quad \text{if} \quad \tilde{z}_{qc} \equiv z_{c}^{n} = \rho_{c}^{n} a_{c}^{n}
\]
Semi-discret first-order scheme

\[ m_c \frac{d U_c}{d t} = - \sum_{q \in Q_c} F_{qc} \cdot l_{qc} n_{qc} \]

Gibbs identity

\[ T \, dS = d\varepsilon + p \, d\tau = d\epsilon - \underline{u} \cdot d\underline{u} + p \, d\tau \]

Semi-discret production of entropy

\[ m_c \, T_c \frac{d S_c}{d t} = m_c \frac{d e_c}{d t} + \underline{u}_c \cdot m_c \frac{d \underline{u}_c}{d t} + p_c \, m_c \frac{d \tau_c}{d t} \]

\[ m_c \, T_c \frac{d S_c}{d t} = \sum_{q \in Q_c} \tilde{z}_{qc} \, l_{qc} \left[ (\overline{u}_q - \underline{u}_c) \cdot \underline{n}_{qc} \right]^2 \geq 0 \]

Positivity of the discrete scheme

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2. Gas dynamics system of equations

3. First-order numerical scheme for the 2D gas dynamics

4. High-order extension in the 2D case

5. Numerical results in 2D
High-order extension of the finite-volume scheme

- MUSCL, (W)ENO, DG, ...

Mean values equation

\[ U_{c}^{n+1} = U_{c}^{n} - \frac{\Delta t^{n}}{m_{c}} \sum_{q \in Q_{c}} \bar{F}_{qc} \cdot l_{qc} n_{qc} \]

- In \( \bar{F}_{qc} \), the mean values are substituted by the high-order values \( U_{qc} \) in \( \omega_{c} \) at points \( q \)

Updated or total Lagrangian formulation

\[ \rho \frac{dU}{dt} + \nabla x \cdot F(U) = 0 \quad \text{ou} \quad \rho^{0} \frac{dU}{dt} + \nabla x \cdot (|J| J^{-1} F(U)) = 0 \]

Piecewise polynomial approximation

- \( U_{h,c}^{n}(x) \) the polynomial approximation of the solution on \( \omega_{c}^{n} \)
- \( U_{h,c}^{n}(X) \) the polynomial approximation of the solution on \( \omega_{c}^{0} \)
- \( U_{qc} = U_{h,c}^{n}(x_{q}) \) (moving config.) or \( U_{qc} = U_{h,c}^{n}(X_{q}) \) (fixed config.)
Numerical results in 2D

2nd order scheme

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Numerical results in 2D
Numerical results in 2D  
2nd order scheme

**Sedov point blast problem**

(a) Pressure field

(b) Density profiles

**Figure:** Solution at time $t = 1$ for a Sedov problem on a $30 \times 30$ Cartesian mesh
Sedov point blast problem

Figure: Solution at time $t = 1$ for a Sedov problem on a $30 \times 30$ Cartesian mesh
Sedov point blast problem

(c) Triangular grid - 1110 cells
(d) Polygonal grid - 775 cells

Figure: Initial unstructured grids for Sedov point blast problem
Sedov point blast problem

Figure: Solution at time $t = 1$ for a Sedov problem on a grid made of 1110 triangular cells
**Sedov point blast problem**

![Density field](image)

![Density profiles](image)

**Figure**: Solution at time $t = 1$ for a Sedov problem on a grid made of 775 polygonal cells
Underwater TNT explosion

Figure: Solution at time $t = 2.5 \times 10^{-4}$ for a underwater TNT explosion on a $120 \times 9$ polar mesh
Underwater TNT explosion

Figure: Solution at time $t = 2.5 \times 10^{-4}$ for a underwater TNT explosion on a $120 \times 9$ polar mesh
Aluminium projectile impact problem

Figure: Solution at time $t = 0.05$ for a projectile impact problem on a $100 \times 10$ Cartesian mesh
Aluminium projectile impact problem

Figure: Solution at time $t = 0.05$ for a projectile impact problem on a $100 \times 10$ Cartesian mesh
Taylor-Green vortex

Figure: Final deformed grids at time $t = 0.75$, on a $10 \times 10$ Cartesian mesh
Taylor-Green vortex

Figure: Final deformed grids at time $t = 0.75$, on a $10 \times 10$ Cartesian mesh
### Convergence rates

<table>
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<tr>
<th>$h$</th>
<th>$E_{L_1}^h$</th>
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<th>$E_{L_2}^h$</th>
<th>$q_{L_2}^h$</th>
<th>$E_{L_\infty}^h$</th>
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Table: Convergence rates on the pressure for a 2nd order DG scheme.
Taylor-Green vortex

Figure: Final deformed grids at time $t = 0.75$, on a $10 \times 10$ Cartesian mesh
Taylor-Green vortex

Figure: Final deformed grids at time $t = 0.75$, on a $10 \times 10$ Cartesian mesh
Convergence rates

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<th>$E_{L_1}^h$</th>
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Table: Convergence rates on the pressure for a 3rd order DG scheme
Polar meshes - symmetry preservation

Figure: Curvilinear grids defined in polar coordinates
Sod shock tube problem - symmetry preservation

Figure: Density fields with 1st and 2nd order schemes on a 3rd mesh
**Sod shock tube problem - symmetry preservation**

**Figure**: 3rd order solution for a Sod shock tube problem on a $100 \times 3$ polar grid
Sod shock tube problem - symmetry preservation

Figure: 3rd order solution for a Sod shock tube problem on a 100 × 1 polar grid
Sod shock tube problem - symmetry preservation

Figure: 3rd order solution for a Sod shock tube problem on a $100 \times 1$ polar grid
Gresho-like vortex problem

Figure: Final deformed grids at time $t = 1$, on a $20 \times 18$ polar mesh
Gresho-like vortex problem

Figure: Final deformed grids at time $t = 1$, on a $20 \times 18$ polar mesh
Figure: Final deformed grids at time $t = 1$, on a $20 \times 18$ polar mesh.
Numerical results in 2D

Gresho-like vortex problem

Figure: Final deformed grids at time $t = 1$, on a $20 \times 18$ polar mesh
Gresho-like vortex problem

Figure: Velocity and pressure profiles at time $t = 1$, on a $20 \times 18$ polar grid
Gresho-like vortex problem

Figure: Density profiles at time $t = 1$, on a $20 \times 18$ polar grid
Kidder isentropic compression

Figure: Initial and final grids for a Kidder problem on a $10 \times 5$ polar mesh
Kidder isentropic compression

Figure: Interior and exterior shell radii evolution for a Kidder problem on a $10 \times 5$ polar mesh
Kidder isentropic compression

(i) Initial and final grids

(j) Shell radii evolution

Figure: 3rd order solution for a Kidder compression problem on a 10 × 3 polar grid
### Numerical results in 2D

#### 3rd order scheme

Accuracy and computational time for a Taylor-Green vortex

<table>
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<tr>
<th>D.O.F</th>
<th>$N$</th>
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**Table: 1st order scheme**

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**Table: 2nd order scheme**

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<tr>
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**Table: 3rd order scheme**


Taylor-Green vortex

Figure: Final deformed grids at time $t = 0.6$, for 16 triangular cells meshes
Taylor-Green vortex

Figure: Final deformed grids at time $t = 0.6$, for 16 triangular cells meshes
Sod shock tube problem - symmetry preservation

Figure: 4th order solution for a Sod shock tube problem on a polar grid made of 308 triangular cells.
Numerical results in 2D  
3rd order scheme

Sedov point blast problem - spurious deformations

Figure: Third-order solution at time $t = 1$ for a Sedov problem on a $30 \times 30$ Cartesian mesh.