Cell-centered discontinuous Galerkin scheme for Lagrangian hydrodynamics

F. Vilar\textsuperscript{1}, P. H. Maire\textsuperscript{1}, R. Abgrall\textsuperscript{2}

\textsuperscript{1}CEA CESTA, BP 2, 33 114 Le Barp, France

\textsuperscript{2}INRIA and University of Bordeaux, Team Bacchus,
Institut de Mathématiques de Bordeaux,
351 Cours de la Libération, 33 405 Talence Cedex, France

September 2011
1 Introduction
   • Discontinuous Galerkin (DG)
     • Scalar conservation laws
     • 1D Lagrangian hydrodynamics

2 2D Lagrangian hydrodynamics
   • References
   • System and equations
   • Geometric consideration
   • 2nd order Deformation tensor
   • 2nd order DG scheme

3 Conclusion
• extension of finite volumes method
• polynomial approximation of the solution in the cells
• high order scheme, high precision

• local variational formulation
• choice of the numerical fluxes (global $L^2$ stability, entropic inequality)
• time discretization - TVD multistep Runge-Kutta

1 Introduction
- Discontinuous Galerkin (DG)
- Scalar conservation laws
- 1D Lagrangian hydrodynamics

2 2D Lagrangian hydrodynamics
- References
- System and equations
- Geometric consideration
- 2nd order Deformation tensor
- 2nd order DG scheme

3 Conclusion
comparison between the second order and the third order scheme with limitation

Figure: linear advection of a combination of smooth and discontinuous profiles
advection : solid body rotation

Burgers

numerical solutions using third order limited DG on a polygonal grid made of 2500 cells
rate of convergence with and without the slope limitation

<table>
<thead>
<tr>
<th>Linear advection</th>
<th>( L_1 )</th>
<th>( L_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>first order</td>
<td>1.02</td>
<td>1.02</td>
</tr>
<tr>
<td>second order</td>
<td>1.99</td>
<td>1.98</td>
</tr>
<tr>
<td>second order lim</td>
<td>2.15</td>
<td>2.15</td>
</tr>
<tr>
<td>third order</td>
<td>2.98</td>
<td>2.98</td>
</tr>
<tr>
<td>third order lim</td>
<td>3.45</td>
<td>3.22</td>
</tr>
</tbody>
</table>

Table: for the smooth solution \( u_0(x) = \sin(2\pi x)\sin(2\pi y) \) on a \([0, 1]^2\) Cartesian grid
<table>
<thead>
<tr>
<th>Introduction</th>
<th>2D Lagrangian hydrodynamics</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discontinuous Galerkin (DG)</td>
<td>Scalar conservation laws</td>
<td>1D Lagrangian hydrodynamics</td>
</tr>
</tbody>
</table>

1. **Introduction**
   - Discontinuous Galerkin (DG)
   - Scalar conservation laws
   - **1D Lagrangian hydrodynamics**

2. **2D Lagrangian hydrodynamics**
   - References
   - System and equations
   - Geometric consideration
   - 2nd order Deformation tensor
   - 2nd order DG scheme

3. **Conclusion**
influence of the limitation on the linearized Riemann invariants

Figure: third order DG for the Sod shock tube problem using 100 cells: density
**3rd order DG scheme with limitation: density**

(a) Shu oscillating shock tube

(b) uniformly accelerated piston

rate of convergence with and without the slope limitation

<table>
<thead>
<tr>
<th></th>
<th>$L_1$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>gas dynamics</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>first order</td>
<td>0.80</td>
<td>0.73</td>
</tr>
<tr>
<td>second order</td>
<td>2.25</td>
<td>2.26</td>
</tr>
<tr>
<td>second order lim</td>
<td>2.04</td>
<td>2.21</td>
</tr>
<tr>
<td>third order</td>
<td>3.39</td>
<td>3.15</td>
</tr>
<tr>
<td>third order lim</td>
<td>2.75</td>
<td>2.72</td>
</tr>
</tbody>
</table>

Table: for a smooth solution in the special case $\gamma = 3$

### Introduction
- Discontinuous Galerkin (DG)
- Scalar conservation laws
- 1D Lagrangian hydrodynamics

### 2D Lagrangian hydrodynamics
- References
- System and equations
- Geometric consideration
- 2nd order Deformation tensor
- 2nd order DG scheme

### Conclusion

September 2011
François Vilar


# Introduction
- Discontinuous Galerkin (DG)
- Scalar conservation laws
- 1D Lagrangian hydrodynamics

## 2D Lagrangian hydrodynamics
- References
- System and equations
  - Geometric consideration
  - 2nd order Deformation tensor
  - 2nd order DG scheme

## Conclusion
gas dynamics system in Lagrangian formalism

\[
\begin{align*}
\rho^0 \frac{d}{dt} \left( \frac{1}{\rho} \right) - \nabla_X \cdot (JF^{-1} U) &= 0 \\
\rho^0 \frac{dU}{dt} + \nabla_X \cdot (JF^{-t} P) &= 0 \\
\rho^0 \frac{dE}{dt} + \nabla_X \cdot (JF^{-1} PU) &= 0
\end{align*}
\]

where \( X \) is the Lagrangian (initial) coordinate

\( F = \frac{\partial x}{\partial X} \) is called the deformation gradient tensor, where \( x \) is the Eulerian (actual) coordinate and \( J = \det(F) \)

using the trajectory equation \( \frac{dx}{dt} = U(x, t) \iff \frac{dF}{dt} = \nabla_X U \) (2)

Piola compatibility condition \( \nabla_X \cdot (JF^{-t}) = 0 \) (3)
1 Introduction
- Discontinuous Galerkin (DG)
- Scalar conservation laws
- 1D Lagrangian hydrodynamics

2 2D Lagrangian hydrodynamics
- References
- System and equations
- Geometric consideration
- 2nd order Deformation tensor
- 2nd order DG scheme

3 Conclusion
being given a mapping \( \mathbf{x} = \Phi(\mathbf{X}, t) \)

\[
F = \nabla_x \Phi
\]  

developing \( \Phi \) on the basis functions \( \lambda_p \) in the cell \( \Omega_c \)

\[
\Phi_h^c(\mathbf{X}, t) = \Phi_h(\mathbf{X}, t)|_{\Omega_c} = \sum_p \lambda_p(\mathbf{X}) \Phi_p(t)
\]

where the \( p \) points are some control points

by setting \( G_c = (JF^{-t})_c \)

\[
\nabla_x \cdot G_c = \sum_p \left( \begin{array}{c} \Phi^y_p(\partial_{yX} \lambda_p - \partial_{yY} \lambda_p) \\ -\Phi^x_p(\partial_{yX} \lambda_p - \partial_{xY} \lambda_p) \end{array} \right) = 0
\]
using (4) and \[ \frac{d}{dt} \Phi_p = U_p \implies \frac{d}{dt} F_c = \sum_p U_p \otimes \nabla x \lambda_p \] (5)

in 2D, \( F \rightarrow JF^{-t} = G \) is a linear function

\( JF^{-t}N \) represents the geometric normal in the Eulerian frame thanks to Nanson formula \( JF^{-t}N dS = GN dS = n ds \)

to ensure this quantity to be continuous, we discretize \( F \) by means of mapping defined on triangular cells \( T^c_i \) with \( i = 1 \ldots ntri \), using finite elements polynomial basis

using the fact \( \frac{d}{dt} F = \nabla x U \), \( F \) approximation order has to be one less than the one obtain with the DG scheme on \( \frac{1}{\rho} \), \( U \) and \( E \)
1 Introduction
- Discontinuous Galerkin (DG)
- Scalar conservation laws
- 1D Lagrangian hydrodynamics

2 2D Lagrangian hydrodynamics
- References
- System and equations
- Geometric consideration
- 2nd order Deformation tensor
- 2nd order DG scheme

3 Conclusion
- for the \( P_1 \) representation, the chosen finite elements polynomial basis in a general triangle \( T_c \) write

\[
\lambda_p(X) = \frac{1}{2|T_c|} [X(Y_{p+} - Y_{p-}) - Y(X_{p+} - X_{p-}) + X_{p+} Y_{p-} - X_{p-} Y_{p+}] \tag{6}
\]

- we can access to \( \nabla_X \lambda_p \) needed in (5)

\[
\nabla_X \lambda_p(X) = \frac{1}{2|T_c|} \left( \begin{array}{c} Y_{p+} - Y_{p-} \\ X_{p-} - X_{p+} \end{array} \right) = \frac{1}{|T_c|} L_{pc} N_{pc} \tag{7}
\]

where \( L_{pc} N_{pc} = L_{p-p} N_{p-p} + L_{pp+} N_{pp+} \)

\[
= - \frac{L_{p+p} N_{p+p}}{2}
\]
the equation (5) rewrites
\[
\frac{d}{dt} F_c = \frac{1}{|T_c|} \sum_{p \in \mathcal{P}(T_c)} U_p \otimes L_{pc} N_{pc} \tag{8}
\]

with this definition, GN continuity is well preserved at the interface be tween triangles

\[
G_c L_{pp^+} N_{pp^+} = \frac{1}{|T_c|} \sum_{p_t \in \mathcal{P}(T_c)} L_{ptc} L_{pp^+} \left( \Phi^Y_p (N^X_{pp^+} N^Y_{ptc} - N^Y_{pp^+} N^X_{ptc}) - \Phi^X_p (N^X_{pp^+} N^Y_{ptc} - N^Y_{pp^+} N^X_{ptc}) \right)
\]

\[
= \frac{1}{|T_c|} \sum_{p_t \in \mathcal{P}(T_c)} \left( L_{pp^+} T_{pp^+} \cdot L_{ptc} N_{ptc} \right) \left( \begin{array}{c} \Phi^Y_p \\ -\Phi^X_p \end{array} \right)
\]

\[
= \left( \begin{array}{c} \Phi^Y_{p^+} - \Phi^Y_p \\ \Phi^X_{p^+} - \Phi^X_p \end{array} \right) = \left( \begin{array}{c} y_{p^+} - y_p \\ x_{p^+} - x_p \end{array} \right) = l_{pp^+} n_{pp^+} \tag{9}
\]
1 Introduction
   - Discontinuous Galerkin (DG)
   - Scalar conservation laws
   - 1D Lagrangian hydrodynamics

2 2D Lagrangian hydrodynamics
   - References
   - System and equations
   - Geometric consideration
   - 2nd order Deformation tensor
   - 2nd order DG scheme

3 Conclusion
Discontinuous Galerkin

- \( \{ \sigma_k^c \}_{k=0}^{K} \) basis of \( \mathbb{P}^{\text{order}-1}(\Omega_c) \)
- \( \phi_h^c(X, t) = \sum_{k=0}^{K} \phi_k^c(t) \sigma_k^c(X) \) approximate of \( \phi(X, t) \) on \( \Omega_c \)
- Taylor basis, \( k_1 + k_2 = k \)

\[
\sigma_k^c = \frac{1}{k_1!k_2!} \left[ \left( \frac{X - X_c}{\Delta X_c} \right)^{k_1} \left( \frac{Y - Y_c}{\Delta Y_c} \right)^{k_2} - \langle \left( \frac{X - X_c}{\Delta X_c} \right)^{k_1} \left( \frac{Y - Y_c}{\Delta Y_c} \right)^{k_2} \rangle \right]
\]

- for the second order scheme, \( K = 2 \)

\[
\sigma_0^c = 1, \quad \sigma_1^c = \frac{X - X_c}{\Delta X_c}, \quad \sigma_2^c = \frac{Y - Y_c}{\Delta Y_c}
\]

where \( \Delta X_c = \frac{X_{\text{max}} - X_{\text{min}}}{2} \) and \( \Delta Y_c = \frac{Y_{\text{max}} - Y_{\text{min}}}{2} \) with \( X_{\text{max}}, Y_{\text{max}}, X_{\text{min}}, Y_{\text{min}} \) the maximum and minimum coordinates in the cell \( \Omega_c \)
Density

- local variational formulation of (1a) on $\Omega_c$

$$
\int_{\Omega_c} \rho^0 \frac{d}{dt} \left( \frac{1}{\rho} \right) \sigma_q d\Omega = \sum_{k=0}^{K} \frac{d}{dt} \left( \frac{1}{\rho} \right)_k \int_{\Omega_c} \rho^0 \sigma_q \sigma_k d\Omega
$$

$$
= \int_{\Omega_c} \sigma_q \nabla x \cdot (JF^{-1} U) d\Omega
$$

$$
= - \int_{\Omega_c} U \cdot JF^{-t} \nabla x \sigma_q d\Omega + \int_{\partial \Omega_c} \bar{U} \cdot \sigma_q JF^{-t} N dL
$$

- $G_i^c = (JF^{-t})_i^c$ is constant on $T_i^c$ and $\nabla x \sigma_q$ over $\Omega_c$

$$
\int_{\Omega_c} \rho^0 \frac{d}{dt} \left( \frac{1}{\rho} \right) \sigma_q d\Omega = - \sum_{i=1}^{ntri} G_i \nabla x \sigma_q \cdot \int_{T_i^c} UdT + \int_{\partial \Omega_c} \bar{U} \cdot \sigma_q G N dL
$$
\[ \int_{\Omega_c} \rho^0 \frac{d}{dt} \left( \frac{1}{\rho} \right) \sigma_q d\Omega \simeq - \sum_{i=1}^{ntri} G_i^c \nabla x \sigma_q \cdot \int_{T_i^c} U d\mathcal{T} \]

\[ + \sum_{p \in P(\Omega_c)} U_p \cdot \int_{\partial \Omega_c \cap \partial \Omega_{pc}} \left\{ \sigma_q G \mathbf{N} dB \right\} \]

\[ l_{pc}^q n_{pc}^q \]

Finally, the equation on the density leads to

\[ \int_{\Omega_c} \rho^0 \frac{d}{dt} \left( \frac{1}{\rho} \right) \sigma_q d\Omega = - \sum_{i=1}^{ntri} G_i^c \nabla x \sigma_q \cdot \int_{T_i^c} U d\mathcal{T} + \sum_{p \in P(\Omega_c)} U_p \cdot l_{pc}^q n_{pc}^q \quad (10) \]

For the first order with \( l_{pc} n_{pc} = l_{pc}^0 n_{pc}^0 \)

\[ m_c \frac{d}{dt} \left( \frac{1}{\rho} \right)_c = \int_{\Omega_c} \rho^0 \frac{d}{dt} \left( \frac{1}{\rho} \right) d\Omega = \sum_{p \in P(\Omega_c)} U_p \cdot l_{pc} n_{pc} \quad (11) \]
Velocity

local variational formulation of (1b) on $\Omega_c$ leads to

$$
\int_{\Omega_c} \rho^0 \frac{d\mathbf{U}}{dt} \sigma_q d\Omega = \sum_{i=1}^{ntri} G_i^c \nabla x \sigma_q \int_{T_i^c} P dT - \sum_{p \in \mathcal{P}(\Omega_c)} F_{pc}^q
$$

where $F_{pc}^q = \int_{\partial \Omega_c \cap \partial \Omega_{pc}} \mathcal{P} \sigma_q G N dL$

for the first order with $F_{pc} = F_{pc}^0$

$$
m_c \frac{d\mathbf{U}_c}{dt} = \int_{\Omega_c} \rho^0 \frac{d\mathbf{U}}{dt} d\Omega = - \sum_{p \in \mathcal{P}(\Omega_c)} F_{pc}
$$
Energy

- local variational formulation of (1c) on $\Omega_c$

$$\int_{\Omega_c} \rho_0 \frac{dE}{dt} \sigma_q d\Omega = \sum_{i=1}^{ntri} G_i^c \nabla X \sigma_q \cdot \int_{T_i^c} P U d\Sigma - \sum_{p \in \mathcal{P}(\Omega_c)} \int_{\partial \Omega_c \cap \partial \Omega_{pc}} P U \cdot \sigma_q G N dL$$  \hspace{1cm} (14)

- we make the following fundamental assumption $P U = \overline{P U}$

- finally, the equation on the energy rewrites

$$\int_{\Omega_c} \rho_0 \frac{dE}{dt} \sigma_q d\Omega = \sum_{i=1}^{ntri} G_i^c \nabla X \sigma_q \cdot \int_{T_i^c} P U d\Sigma - \sum_{p \in \mathcal{P}(\Omega_c)} U_p \cdot F_{pc}^q$$  \hspace{1cm} (15)

- for the first order

$$m_c \frac{dE_c}{dt} = \int_{\Omega_c} \rho_0 \frac{dE}{dt} d\Omega = - \sum_{p \in \mathcal{P}(\Omega_c)} U_p \cdot F_{pc}$$  \hspace{1cm} (16)
The use of variational formulations and Gibbs formula leads to

\[\int_{\Omega_c} \rho^0 T \frac{dS}{dt} d\Omega = \int_{\partial\Omega_c} \left[ \overline{P} \mathbf{U} + P \overline{U} - \overline{P} \mathbf{U} - P \mathbf{U} \right] \cdot \mathbf{G N dL} \]

\[= \sum_{f \in \mathcal{F}(\Omega_c)} \int_f (\overline{P} - P)(\mathbf{U} - \overline{U}) \cdot \mathbf{G N dL} \quad (17)\]

A sufficient condition to satisfy \(\int_{\Omega_c} \rho^0 T \frac{dS}{dt} d\Omega \geq 0\) consists in setting

\[\overline{P}(\mathbf{X}_f) = Pc(\mathbf{X}_f) - Z_c(\overline{U}(\mathbf{X}_f) - U_c(\mathbf{X}_f)) \cdot \frac{\mathbf{G N}}{\|\mathbf{G N}\|} \quad (18)\]

where \(\mathbf{X}_f\) is a point on the face \(f\) and \(Z_c\) a positive constant with a physical dimension of a density times a velocity.
using this expression to calculate $F^q_{pc}$ leads to

\[
F^q_{pc} = \int_{\partial \Omega_c \cap \partial \Omega_{pc}} \overline{P} \sigma_q JF^{-t} N dL
\]

\[
= \int_{\partial \Omega_c \cap \partial \Omega_{pc}} P_c \sigma_q G N dL - \int_{\partial \Omega_c \cap \partial \Omega_{pc}} Z_c (\overline{U} - U_c) \cdot \frac{G N}{\|G N\|} \sigma_q G N dL
\]

\[
\simeq P_c(p) \int_{\partial \Omega_c \cap \partial \Omega_{pc}} \sigma_q G N dL
\]

\[
- \int_{\partial \Omega_c \cap \partial \Omega_{pc}} Z_c (U_p - U_c(p)) \cdot \frac{G N}{\|G N\|} \sigma_q G N dL
\]

finally, $F^q_{pc}$ writes

\[
F^q_{pc} = P_c(p) I^q_{pc} n^q_{pc} - M^q_{pc} (U_p - U_c(p))
\]
\[ M_{pc}^q \] are defined as \[ M_{pc}^q = Z_c \int_{\partial \Omega_c \cap \partial \Omega_{pc}} \frac{G_N}{||G_N||} \otimes G_N \sigma_q dL \]

\[ = Z_c (l_{pc}^{q,+} n_{pc}^+ \otimes n_{pc}^+ + l_{pc}^{q,-} n_{pc}^- \otimes n_{pc}^-) \]

where \[ l_{pc}^{q,\pm} = \int_{\partial \Omega_c \cap \partial \Omega_{pc}^\pm} \sigma_q dL \]

\[ M_{pc}^0 = M_{pc} = Z_c (l_{pc}^{+,} n_{pc}^+ \otimes n_{pc}^+ + l_{pc}^{-} n_{pc}^- \otimes n_{pc}^-) \] is semi definite positive matrix with a physical dimension of a density times a velocity

to be conservative in total energy over the whole domain,

\[ \sum_{c \in C(p)} F_{pc} = 0 \] and consequently

\[ (\sum_{c \in C(p)} M_{pc}) U_p = \sum_{c \in C(p)} [P_c(p) l_{pc} n_{pc} + M_{pc} U_c(p)] \quad (20) \]
Sod shock tube problem on a polar grid made of 500 cells: density map with limitation
expansion wave into vacuum problem on a polar grid made of 250 cells: internal energy map with limitation
Noh problem on a Cartesian grid made of 2500 cells: density map
Sedov problem on a Cartesian grid made of 900 cells and a polygonal one made of 775 cells: density map with limitation
initial grid

actual grid

Gresho problem on a polar grid made of 720 cells: pressure map with limitation
discontinuous Galerkin

discontinuous Galerkin limited

Taylor-Green vortex problem on a cartesian grid made of 400 cells: pressure map without limitation at t=0.75s
### Table: rate of convergence computed for second order DG scheme

<table>
<thead>
<tr>
<th>$h$</th>
<th>$q_h^{L_2}$</th>
<th>$q_h^{L_\infty}$</th>
<th>$q_h^{L_2}$</th>
<th>$q_h^{L_\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{20}$</td>
<td>1.74</td>
<td>1.35</td>
<td>2.05</td>
<td>1.54</td>
</tr>
<tr>
<td>$\frac{1}{40}$</td>
<td>1.85</td>
<td>1.85</td>
<td>2.11</td>
<td>1.81</td>
</tr>
<tr>
<td>$\frac{1}{80}$</td>
<td>1.42</td>
<td>2.34</td>
<td>1.58</td>
<td>1.54</td>
</tr>
</tbody>
</table>

### Table: numerical errors computed at t=0.6s on the pressure

<table>
<thead>
<tr>
<th>$h$</th>
<th>$E_h^{L_2}$</th>
<th>$E_h^{L_\infty}$</th>
<th>$E_h^{L_2}$</th>
<th>$E_h^{L_\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{20}$</td>
<td>1.854E-2</td>
<td>6.596E-2</td>
<td>1.120E-2</td>
<td>3.678E-2</td>
</tr>
<tr>
<td>$\frac{1}{40}$</td>
<td>6.500E-3</td>
<td>2.452E-2</td>
<td>3.356E-3</td>
<td>1.446E-2</td>
</tr>
<tr>
<td>$\frac{1}{80}$</td>
<td>1.817E-3</td>
<td>9.122E-3</td>
<td>9.314E-4</td>
<td>4.019E-3</td>
</tr>
<tr>
<td>$\frac{1}{160}$</td>
<td>4.944E-4</td>
<td>2.555E-3</td>
<td>3.471E-4</td>
<td>7.959E-4</td>
</tr>
</tbody>
</table>

Table: numerical errors computed at t=0.6s on the pressure
1 Introduction
   - Discontinuous Galerkin (DG)
   - Scalar conservation laws
   - 1D Lagrangian hydrodynamics

2 2D Lagrangian hydrodynamics
   - References
   - System and equations
   - Geometric consideration
   - 2nd order Deformation tensor
   - 2nd order DG scheme

3 Conclusion
Conclusions

- DG schemes up to 3rd order
  - linear and non-linear scalar conservation laws in 1D and 2D on general unstructured grids
  - 1D gas dynamics system in Lagrangian formalism
- DG scheme up to 2nd order for the 2D gas dynamics system in Lagrangian formalism with particular geometric consideration
- numerical flux studies
- Riemann invariants limitation

Prospects

- 3rd order DG scheme for the 2D gas dynamics system in Lagrangian formalism
- validation
- extension to ALE