Positivity-preserving cell-centered Lagrangian schemes

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2. **Lagrangian and Eulerian descriptions**
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2. Lagrangian and Eulerian descriptions

3. Compatible first-order positivity-preserving discretization

4. High-order positivity-preserving extension

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### Finite volume schemes on moving mesh
- J. K. Dukowicz: CAVEAT scheme, 1986
- B. Després: GLACE scheme, 2005
- G. Kluth: Cell-centered Lagrangian scheme for the hyperelasticity, 2010
- S. Del Pino: Curvilinear finite-volume Lagrangian scheme, 2010
- P. Hoch: Finite volume method on unstructured conical meshes, 2011

### DG scheme on initial mesh
- R. Loubère: DG scheme for Lagrangian hydrodynamics, 2004
- Z. Jia: DG spectral finite element for Lagrangian hydrodynamics, 2010
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1. Cell-Centered Lagrangian schemes

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Flow transformation of the fluid

- The fluid flow is described mathematically by the continuous transformation, \( \Phi \), so-called mapping such as \( \Phi : X \longrightarrow x = \Phi(X, t) \)

![Flow map diagram](image)

**Figure**: Notation for the flow map.

where \( X \) is the Lagrangian (initial) coordinate, \( x \) the Eulerian (actual) coordinate, \( N \) the Lagrangian normal and \( n \) the Eulerian normal.

**Deformation Jacobian matrix: deformation gradient tensor**

- \( F = \nabla_X \Phi = \frac{\partial x}{\partial X} \) and \( J = \det F > 0 \)
Trajectory equation

\[ \frac{d \mathbf{x}}{dt} = \mathbf{U}(\mathbf{x}, t), \quad \mathbf{x}(\mathbf{X}, 0) = \mathbf{X} \]

Material time derivative

\[ \frac{d}{dt} f(\mathbf{x}, t) = \frac{\partial}{\partial t} f(\mathbf{x}, t) + \mathbf{U} \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, t) \]

Transformation formulas

- \( \text{FdX} = \mathbf{dX} \)
- \( \rho^0 = \rho J \)
- \( JdV = dV \)
- \( JF^{-1} \mathbf{N} dS = \mathbf{n} ds \)

Differential operators transformations

- \( \nabla_{\mathbf{x}} P = \frac{1}{J} \nabla_{\mathbf{x}} \cdot (P JF^{-1}) \)
- \( \nabla_{\mathbf{x}} \cdot \mathbf{U} = \frac{1}{J} \nabla_{\mathbf{x}} \cdot (JF^{-1} \mathbf{U}) \)
### Piola compatibility condition

\[ \nabla_x (JF^{-t}) = 0 \quad \Rightarrow \quad \int_{\Omega} \nabla_x (JF^{-t}) \, dV = \int_{\partial \Omega} JF^{-t} N \, dS = \int_{\partial \omega} n \, ds = 0 \]

### Deformation gradient tensor

\[ \frac{dF}{dt} - \nabla_x U = 0 \]

### Actual configuration

\[ \rho \frac{d}{dt} \left( \frac{1}{\rho} \right) - \nabla_x U = 0 \]

\[ \rho \frac{dU}{dt} + \nabla_x P = 0 \]

\[ \rho \frac{de}{dt} + \nabla_x (PU) = 0 \]

### Initial configuration

\[ \rho^0 \frac{d}{dt} \left( \frac{1}{\rho} \right) - \nabla_x (JF^{-1} U) = 0 \]

\[ \rho^0 \frac{dU}{dt} + \nabla_x (P JF^{-t}) = 0 \]

\[ \rho^0 \frac{de}{dt} + \nabla_x (JF^{-1} PU) = 0 \]

### Specific internal energy

\[ \varepsilon = e - \frac{1}{2} U^2 \]
Ideal EOS for the perfect gas

- \( P = \rho (\gamma - 1) \varepsilon \) where \( a = \sqrt{\frac{\gamma P}{\rho}} \)
- If \( \rho > 0 \) then \( \varepsilon > 0 \iff a^2 > 0 \iff P > 0 \)

Stiffened EOS for water

- \( P = \rho (\gamma - 1) \varepsilon - \gamma P^* \) where \( a = \sqrt{\frac{\gamma (P + P^*)}{\rho}} \)
- If \( \rho > 0 \) then \( \rho \varepsilon > P^* \iff a^2 > 0 \iff P > -P^* \)

Jones-Wilkins-Lee (JWL) EOS for the detonation-products gas

- \( P = \rho (\gamma - 1) \varepsilon + f(\rho) \) where \( a = \sqrt{\frac{\gamma P - f(\rho) + \rho f'(\rho)}{\rho}} \)
- If \( \rho > 0 \) then \( \varepsilon > 0 \implies a^2 > 0 \iff P > f(\rho) \geq 0 \)
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**Mass averaged values equations**

- \( m_c \left( \frac{1}{\rho} \right)_c^{n+1} = m_c \left( \frac{1}{\rho} \right)_c^n + \Delta t \sum_{p \in Q(\partial \omega_c)} U_p^n \cdot l_{pc}^n n_{pc}^n \)
- \( m_c U_c^{n+1} = m_c U_c^n - \Delta t \sum_{p \in Q(\partial \omega_c)} F_{pc}^n \)
- \( m_c e_c^{n+1} = m_c e_c^n - \Delta t \sum_{p \in Q(\partial \omega_c)} U_p^n \cdot F_{pc}^n \)

**Definitions**

- \( \psi_c = \frac{1}{m_c} \int_{\Omega_c} \rho^0 \psi \, dV = \frac{1}{m_c} \int_{\omega_c} \rho \psi \, dV \) mean value
- \( F_{pc} = P_c l_{pc} n_{pc} - M_{pc} (U_p - U_c) \) subcell forces

**Momentum and total energy conservation**

- \[ \sum_{c \in C(p)} F_{pc} = 0 \quad \Rightarrow \quad (\sum_{c \in C(p)} M_{pc}) U_p = \sum_{c \in C(p)} (P_c l_{pc} n_{pc} + M_{pc} U_c) \]
**GLACE assumptions**

\( a) \ Q(\partial \omega_c) = \mathcal{P}(\omega_c) \) the node set

\( b) \ l_{pc}n_{pc} = l_{pc}n_{pc}^- + l_{pc}n_{pc}^+ = \frac{1}{2} l_{p^-}n_{p^-} + \frac{1}{2} l_{pp^+}n_{pp^+} \)

\( c) \ M_{pc} = Z_{pc} l_{pc}n_{pc} \otimes n_{pc} \)

\( d) \ U_p = (\sum_{c \in C(p)} M_{pc})^{-1} \sum_{c \in C(p)} (P_c l_{pc}n_{pc} + M_{pc} U_c) \)

**EUCCLHYD assumptions**

- Same assumptions \( a), b) and d) as GLACE

\( c) \ M_{pc} = Z_{pc} l_{pc}n_{pc}^- \otimes n_{pc}^- + Z_{pc} l_{pc}n_{pc}^+ \otimes n_{pc}^+ \)
### Cell-centered DG (CCDG) assumptions

#### a) $Q(\partial \omega_c) = \bigcup_{p \in \mathcal{P}(\omega_c)} (Q(pp^+) \setminus \{p^+\})$

#### b) For $q \in Q(pp^+)$, 

$$l_q n_{q|pp} = \int_0^1 \lambda_q(\zeta) \sum_{k \in Q(pp^+)} \frac{\partial \lambda_k}{\partial \zeta} (x_k \times e_z) \ d\zeta$$

For $p \in \mathcal{P}(\omega_c)$, 

$$l_{pc} n_{pc} = l_p n_{p|p-} + l_p n_{p|pp+}$$

For $q \in Q(pp^+) \setminus \{p, p^+\}$, 

$$l_{qc} n_{qc} = l_q n_{q|pp+}$$
CCDG assumptions

c) For $p \in \mathcal{P}(\omega_c)$,  
$$M_{pc} = Z_{pc}^{-} l_{pc}^{-} n_{pc}^{-} \otimes n_{pc}^{-} + Z_{pc}^{+} l_{pc}^{+} n_{pc}^{+} \otimes n_{pc}^{+}$$

For $q \in \mathcal{Q}(pp^{+}) \setminus \{p, p^{+}\}$,  
$$M_{pc} = Z_{pc} l_{pc} n_{pc} \otimes n_{pc}$$

d) For $p \in \mathcal{P}(\omega_c)$,  
$$U_p = \left( \sum_{c \in \mathcal{C}(p)} M_{pc} \right)^{-1} \sum_{c \in \mathcal{C}(p)} \left( P_c l_{pc} n_{pc} + M_{pc} U_c \right)$$

For $q \in \mathcal{Q}(pp^{+}) \setminus \{p, p^{+}\}$,  
$$U_p = \frac{Z_{pL} U_L + Z_{pR} U_R}{Z_{pL} + Z_{pR}} - \frac{P_R - P_L}{Z_{pL} + Z_{pR}} n_{pL}$$
### CFL condition

- **System eigenvalues:** \(-a, 0, a\)

\[
\forall c, \quad \Delta t \leq C_e \frac{v_c^n}{a_c L_c}
\]

### Volume control

- **Relative volume variation:**

\[
\frac{|v_c^{n+1} - v_c^n|}{v_c^n} \leq C_v
\]

\[
\forall c, \quad \Delta t \leq C_v \frac{v_c^n}{\sum_{p \in Q(\partial \omega_c)} |\sum_{l_p \in n_{pc}} U_p^n \cdot l_p^n n_{pc}^n|}
\]
Solution vector

- \( W = (\frac{1}{\rho}, U, e)^t \)

Admissible convex set

- \( G = \{ W, \rho > 0, \varepsilon = e - \frac{1}{2}U^2 > 0 \} \) for ideal and JWL EOS
- \( G = \{ W, \rho > 0, \varepsilon = e - \frac{1}{2}U^2 > \frac{P^*}{\rho} \} \) for stiffened EOS

First-order positivity-preserving scheme

- If \( W^n_c = ((\frac{1}{\rho})^n_c, U^n_c, e^n_c)^t \in G \), then under which constraint \( W^{n+1}_c \in G \)?

Positive density

- If \( (\frac{1}{\rho})^n_c > 0 \) then \( (\frac{1}{\rho})^{n+1}_c > 0 \iff (\frac{1}{\rho})^n_c > -\frac{\Delta t}{m_c} \sum_{\rho \in Q(\partial \omega_c)} U^n_p \cdot l^n_{pc} n^n_{pc} \)
- Thus if \( C_v < 1 \) then \( (\frac{1}{\rho})^n_c = \frac{v^n_c}{m_c} > 0 \iff (\frac{1}{\rho})^{n+1}_c = \frac{v^{n+1}_c}{m_c} > 0 \)
Positive internal energy

\[ \varepsilon_c = e_c - \frac{1}{2}(U_c)^2 \]
\[ \varepsilon_{c}^{n+1} = \varepsilon_{c}^{n} - \frac{\Delta t}{m_c} \left( \sum_p U^n_p \cdot F^n_{pc} - \sum_p U^n_c \cdot F^n_{pc} + \frac{\Delta t}{2m_c} \left( \sum_p F^n_{pc} \right)^2 \right) \]

Properties

\[ F_{pc} = P_c \ l_{pc} \ n_{pc} - M_{pc}(U_p - U_c) \]
\[ \sum_{p \in Q(\partial \omega_c)} l_{pc} n_{pc} = \sum_{p \in P(\omega_c)} l_{pp^+} n_{pp^+} = 0 \]

Definitions

\[ \lambda_c = \frac{\Delta t}{m_c} \]
\[ V_p = U^n_p - U^n_c \]
CCLS Descriptions
1st order
High-order
CCDG numerical results
Conclusion
Schemes
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Positivity
Stability

Definitions

- $\varepsilon_{c}^{n+1} = A_c + \lambda_c B_c$
- $A_c = \varepsilon_{c}^{n} - \frac{P_{c}^{n}}{\rho_{c}^{n}} \frac{V_{c}^{n+1} - V_{c}^{n}}{V_{c}^{n}}$
- $B_c = \sum_{p} M_{pc} \mathbf{V}_p \cdot \mathbf{V}_p - \frac{\lambda_c}{2} (\sum_{p} M_{pc} \mathbf{V}_p)^2$

$A_c > 0$ for ideal and JWL EOS

- If $B_c \geq 0$ then $A_c > 0 \implies \varepsilon_{c}^{n+1} > 0$
- As $\rho_{c}^{n} > 0$ and $\varepsilon_{c}^{n} > 0$ then $A_c > \varepsilon_{c}^{n} - \frac{P_{c}^{n}}{\rho_{c}^{n}} C_v$

Thus $C_v < \frac{\rho_{c}^{n} \varepsilon_{c}^{n}}{P_{c}^{n}} = \left\{ \begin{array}{ll} 1 & \text{for ideal gas} \\ \frac{1}{\gamma - 1} & \text{for JWL gas} \\ \gamma - 1 + \frac{f(\rho_{c}^{n})}{\rho_{c}^{n} \varepsilon_{c}^{n}} & \end{array} \right.$

$\implies A_c > 0$
\( A_c > \frac{P^*}{\rho_c^{n+1}} \) for stiffened EOS

- If \( B_c \geq 0 \) then \( A_c > \frac{P^*}{\rho_c^{n+1}} \implies \varepsilon_c^{n+1} > \frac{P^*}{\rho_c^{n+1}} \)

- \( A_c = (\varepsilon_c^n - \frac{P^*}{\rho_c^n}) (1 - (\gamma - 1) \frac{v_c^{n+1} - v_c^n}{v_c^n}) + \frac{P^*}{\rho_c^{n+1}} \)

- Since \( \varepsilon_c^n - \frac{P^*}{\rho_c^n} > 0 \) then \( A_c > (\varepsilon_c^n - \frac{P^*}{\rho_c^n}) (1 - (\gamma - 1) C_v) + \frac{P^*}{\rho_c^{n+1}} \)

- Thus \( C_v < \frac{1}{\gamma - 1} \implies A_c > \frac{P^*}{\rho_c^{n+1}} \)

**Discrete entropy inequality**

\[
\lambda_c B_c = \varepsilon_c^{n+1} - A_c = \varepsilon_c^{n+1} - \varepsilon_c^n + P_c^n \left( \left( \frac{1}{\rho} \right)_c^{n+1} - \left( \frac{1}{\rho} \right)_c^n \right)
\]

**Entropy**

\[
T dS = d\varepsilon + P d\left( \frac{1}{\rho} \right) \geq 0 \quad \text{Gibbs identity + second law of thermodynamics}
\]
\( B_c \geq 0 \)

- \( B_c = \sum_{p \in Q(\partial \omega_c)} M_{pc} \mathbf{V}_p \cdot \mathbf{V}_p - \frac{\lambda_c}{2} \left( \sum_{p \in Q(\partial \omega_c)} M_{pc} \mathbf{V}_p \right)^2 \)

- \( M_{pc} = \sum_{n=1}^{N_p} Z_{pn} l_{pn} \mathbf{n}_{pn} \otimes \mathbf{n}_{pn} \)

- \( \sum_{p \in Q(\partial \omega_c)} M_{pc} \mathbf{V}_p \cdot \mathbf{V}_p = \sum_{p \in Q(\partial \omega_c)} \sum_{n=1}^{N_p} Z_{pn} l_{pn} (\mathbf{V}_p \cdot \mathbf{n}_{pn})^2 = \sum_{p \in Q(\partial \omega_c)} \sum_{n=1}^{N_p} Z_{pn} l_{pn} X_{pn}^2 \)

- Re-numbering: \( \sum_{p \in Q(\partial \omega_c)} \sum_{n=1}^{N_p} \psi_{pn} = \sum_{i=1}^{N_c} \psi_i \)

- \( B_c = \sum_{i=1}^{N_c} Z_i l_i X_i^2 - \frac{\lambda_c}{2} \sum_{i,j=1}^{N_c} Z_i Z_j l_i l_j X_i X_j (\mathbf{n}_i \cdot \mathbf{n}_j) = \mathbf{H} \mathbf{X} \cdot \mathbf{X} \)

where \( \mathbf{X} = (X_1, \ldots, X_{N_c})^t \) and \( H_{ij} = \begin{cases} 
Z_i l_i (1 - \frac{\lambda_c}{2} Z_i l_i), & \text{if } i = j, \\
- \frac{\lambda_c}{2} Z_i Z_j l_i l_j (\mathbf{n}_i \cdot \mathbf{n}_j), & \text{if } i \neq j.
\end{cases} \)
### Theorem

- If $H$ is symmetric diagonally dominant with non-negative diagonal entries then $H$ is positive semi-definite (thanks to Gerschgorin theorem)

### $B_c \geq 0$

1. If $\lambda_c \leq \frac{2}{Z_i l_i}$ then $H_{ii} \geq 0$
2. If $\lambda_c \leq \frac{2}{\sum_j Z_j l_j |n_i \cdot n_j|}$ then $|H_{ii}| - \sum_{j \neq i} |H_{ij}| \geq 0$

Thus if $\lambda_c \leq \frac{2}{\sum_j Z_j l_j} \iff \Delta t \leq \frac{m_c}{\frac{1}{2} \sum_j Z_j l_j}$ then $B_c \geq 0$

### Acoustic impedance $Z_c = \rho_c a_c$

- If $\Delta t \leq \frac{v^n_c}{a_c L_c}$ where $L_c = \frac{1}{2} \sum_j l_j$ then $B_c \geq 0$
Finally, for the first-order finite volume cell-centered Lagrangian schemes, if

1. $W^n_c \in G$

2. $\Delta t \leq C_v \frac{v^n_c}{\sum_{p \in Q(\partial \omega_c)} |U^n_p \cdot l^n_{pc} n^n_{pc}|}$, with $C_v < \min \left(1, \frac{1}{\gamma - 1 + \frac{f(\rho^n_c)}{\rho^n_c \varepsilon^n_c}} \right)$

3. $\Delta t \leq \frac{v^n_c}{a_c L_c}$, with $L_c = \begin{cases} \frac{1}{2} \sum_{p \in P(\omega_c)} l_{pc}, & \text{GLACE} \\ \frac{1}{2} \sum_{p \in P(\omega_c)} l_{pp}, & \text{EUCCLHYD} \\ \frac{1}{2} \sum_{p \in P(\omega_c)} \sum_{q \in Q(pp^+)} l_{q|pp^+}. & \text{CCDG} \end{cases}$

Then $W^{n+1}_c \in G$ and $\varepsilon^{n+1}_c - \varepsilon^n_c + P^n_c \left((1/\rho)_c^{n+1} - (1/\rho)_c^n\right) \geq 0$
Norm definitions

- $\|\psi\|_{L_1} = \int_{\Omega} \rho^0 |\psi| \, dV = \int_{\omega} \rho |\psi| \, dv$

- $\|\psi\|_{L_2} = \left( \int_{\Omega} \rho^0 \psi^2 \, dV \right)^{\frac{1}{2}} = \left( \int_{\omega} \rho \psi^2 \, dv \right)^{\frac{1}{2}}$

Stability analysis

For sake of simplicity periodic boundary conditions (PBC) are considered.

$\psi_h^n$ is the piecewise constant numerical solution such as $\psi_h^n|_{\omega_c} = \psi_c^n$

We assume the initial solution vector $W_c^0 = ((\frac{1}{\rho})_c^0, \mathbf{U}_c^0, e_c^0)^t$ on cell $\omega_c$ is computed through

$$W_c^0 = \frac{1}{m_c} \int_{\Omega_c} \rho^0(\mathbf{X}) \, W^0(\mathbf{X}) \, dV,$$

where $W^0 = (\frac{1}{\rho^0}, \mathbf{U}^0, e^0)^t$ and $\frac{1}{\rho^0}, \mathbf{U}^0, e^0$ respectively are the initial specific volume, velocity and total energy.
Specific volume

- Positivity $|\left(\frac{1}{\rho}\right)_c^n| = \left(\frac{1}{\rho}\right)_c^n$

- Conservation $\sum_c m_c \left(\frac{1}{\rho}\right)_c^n = \sum_c m_c \left(\frac{1}{\rho}\right)_c^{n-1}$ (since PBC + $\sum_{c \in C(p)} l_{pc} n_{pc} = 0$)

$$\|\left(\frac{1}{\rho}\right)_h^n\|_{L_1} = \sum_c m_c |\left(\frac{1}{\rho}\right)_c^n| = \sum_c m_c |\left(\frac{1}{\rho}\right)_c^{n-1}| = \|\left(\frac{1}{\rho}\right)_h^{n-1}\|_{L_1}$$

Total energy

- Positivity $|e^n_c| = e^n_c$ (since $\varepsilon^n_c > 0 \iff e^n_c > \frac{1}{2} (U^n_c)^2 \geq 0$)

- Conservation $\sum_c m_c e^n_c = \sum_c m_c e^{n-1}_c$ (since PBC + $\sum_{c \in C(p)} F_{pc} = 0$)

$$\|e_h^n\|_{L_1} = \sum_c m_c |e^n_c| = \sum_c m_c |e^{n-1}_c| = \|e_h^{n-1}\|_{L_1}$$
Kinetic energy and velocity

- \( K = \frac{1}{2} \mathbf{U}^2 \) specific kinetic energy

- \( \frac{1}{2} (\mathbf{U}_c^n)^2 < e_c^n \quad \Rightarrow \quad \frac{1}{2} \sum_c m_c (\mathbf{U}_c^n)^2 < \sum_c m_c e_c^n \)

- \( 2m_c e_c^n = 2\sqrt{m_c} \sqrt{m_c (e_c^n)^2} \leq m_c + m_c (e_c^n)^2 \)

- \( \sum_c m_c (\mathbf{U}_c^n)^2 < \sum_c m_c + \sum_c m_c (e_c^n)^2 \)

Stability

- \( \| (\frac{1}{\rho})^n_h \|_{L_1} = \| \frac{1}{\rho^0} \|_{L_1} \)

- \( \| e_h^n \|_{L_1} = \| e^0 \|_{L_1} \)

- \( \| K_h^n \|_{L_1} < \| e_h^n \|_{L_1} \)

- \( \| U_h^n \|_{L_2}^2 < m_\omega + \| e_h^n \|_{L_2}^2 \)
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Positivity-preserving cell-centered scheme
Control point solvers

In the control point solvers, $F_{pc}$ and $U_p$, the interpolation values at point $p$ of the high-order approximations of the pressure and velocity, $P_c^h(p)$ and $U_c^h(p)$, are used instead of the mean values $P_c$ and $U_c$.

High-order extension

1. Piecewise linear approximations of the pressure and velocity, $P_h(p)$ and $U_h(p)$, are constructed using the mean values, $P_c$ and $U_c$, over the cells (GLACE and EUCCLHYD).

2. A piecewise polynomial reconstruction of the solution vector $W_h(x) = (\left(\frac{1}{\rho}\right)_h(x), U_h(x), e_h(x))^t$ is assumed, such as its mass averaged value over cell $\omega_c$ corresponds to $W_c$ (CCDG).

The pressure is pointwisely defined through the EOS, such as

$$P_h(x) = \rho_h(x) (\gamma - 1) (e_h(x) - \frac{1}{2}(U_h(x)^2)) + f(\rho_h(x)) - \gamma P^*$$
Quadrature rule over triangles

- Exact for polynomials up to degree $2(d - 1)$
- containing the cell boundary control points, i.e., $Q(\partial \Omega_c) \subset \bigcup_{i=1}^{ntri} R_{i,c}$
- With positive weights, i.e., $\forall q \in R_{i,c}, w_q \geq 0$

GLACE and EUCCLHYD schemes

- $\psi_c = \frac{1}{m_c} \int_{\omega_c} \rho_c \psi_h^c \, dv = \frac{1}{m_c} \sum_{i=1}^{ntri} |\tau_i^c| \sum_{q \in R_{i,c}} w_q \rho_c \psi_h^c(q)$
- $m_q^c = \sum_{i,R_{i,c} \ni q} |\tau_i^c| w_q \rho_c$

CCDG scheme

- $\psi_c = \frac{1}{m_c} \int_{\Omega_c} \rho^0 \psi_h^c \, dV = \frac{1}{m_c} \sum_{i=1}^{ntri} |\tau_i^c| \sum_{q \in R_{i,c}} w_q \rho^0(q) \psi_h^c(q)$
- $m_q^c = \sum_{i,R_{i,c} \ni q} |\tau_i^c| w_q \rho^0(q)$
**Properties**

- \( \mathcal{R}_c = \bigcup_{i=1}^{ntri} \mathcal{R}_{i,c} \)

- \( m_c = \int_{\Omega_c} \rho^0 \, dV = \rho_c \int_{\omega_c} \, d\nu = \sum_{q \in \mathcal{R}_c} m_q \)

- \( \psi_c = \frac{1}{m_c} \sum_{q \in \mathcal{R}_c} m_q \psi_q^c(q) \)

- \( m_c^* = m_c - \sum_{p \in \mathcal{Q}(\partial \omega_c)} m_p \)

- \( \psi_q^c = \frac{1}{m_c^*} \sum_{q \in \mathcal{R}_c \setminus \mathcal{Q}(\partial \omega_c)} m_q \psi_q^c(q) \)

- \( \psi_c = \frac{m_c^*}{m_c} \psi_q^c + \frac{1}{m_c} \sum_{p \in \mathcal{Q}(\partial \omega_c)} m_p \psi_p^c(p) \)
Mass averaged value equations

\[ m_c \left( \frac{1}{\rho} \right)_c^{n+1} = m_c \left( \frac{1}{\rho} \right)_c^n + \Delta t \sum_{p \in Q(\partial \omega_c)} U_p^n \cdot l_{pc}^n n_{pc}^n \]

\[ m_c U_c^{n+1} = m_c U_c^n - \Delta t \sum_{p \in Q(\partial \omega_c)} F_{pc}^n \]

\[ m_c e_c^{n+1} = m_c e_c^n - \Delta t \sum_{p \in Q(\partial \omega_c)} U_p^n \cdot F_{pc}^n \]

Decomposition

\[ \left( \frac{1}{\rho} \right)_c^{n+1} = \frac{m_c}{m_c^*} \left( \frac{1}{\rho} \right)_c^* + \frac{1}{m_c} \sum_{p \in Q(\partial \omega_c)} m_p^c \left( \frac{1}{\rho} \right)_h(p) + \frac{\Delta t}{m_p^c} U_p^n \cdot l_{pc}^n n_{pc}^n \]

\[ U_c^{n+1} = \frac{m_c^*}{m_c} U_c^* + \frac{1}{m_c} \sum_{p \in Q(\partial \omega_c)} m_p^c \left( U_h^c(p) - \frac{\Delta t}{m_p^c} F_{pc}^n \right) \]

\[ e_c^{n+1} = \frac{m_c^*}{m_c} e_c^* + \frac{1}{m_c} \sum_{p \in Q(\partial \omega_c)} m_p^c \left( e_h^c(p) - \frac{\Delta t}{m_p^c} U_p^n \cdot F_{pc}^n \right) \]
Procedure

- Express these equations as a convex combination of first-order schemes


Specific volume

\[
\sum_{p \in Q(\partial \omega)} l_{pc} n_{pc} = 0 \iff l_{pc} n_{pc} = - \sum_{q \in Q(\partial \omega) \setminus p} l_{qc} n_{qc}
\]

\[
h^\rho_p = (\frac{1}{\rho})^c_h(p) + \frac{\Delta t}{m_p^c} U^n_p \cdot l^n_{pc} n^n_{pc}
\]

\[
H^\rho_p = (\frac{1}{\rho})^c_h(p) + \frac{\Delta t}{m_p^c} (U^n_p - V_c) \cdot l^n_{pc} n^n_{pc} = (\frac{1}{\rho})^c_h(p) + \frac{\Delta t}{m_p^c} \sum_{q \in Q(\partial \omega)} V^p_q \cdot l^n_{qc} n^n_{qc}
\]

where \( V^p_q = \begin{cases} U^n_p, & \text{if } p = q, \\ V_c, & \text{if } p \neq q. \end{cases} \)
Momentum

\[ h_p^u = U_h(p) - \frac{\Delta t}{m_p} F_{pc} \]

\[ \sum_{p \in Q(\partial \omega_c)} \mathcal{F}_{pc} = 0 \iff \mathcal{F}_{pc} = - \sum_{q \in Q(\partial \omega_c) \setminus p} \mathcal{F}_{qc} \]

\[ H_p^u = U_h(p) - \frac{\Delta t}{m_p} (F_{pc} - \mathcal{F}_{pc}) = U_h(p) - \frac{\Delta t}{m_p} \sum_{q \in Q(\partial \omega_c)} \mathcal{F}_{q}^p \]

where \[ \mathcal{F}_{q}^p = \begin{cases} F_{pc}, & \text{if } p = q, \\ \mathcal{F}_{qc}, & \text{if } p \neq q. \end{cases} \]

Total energy

\[ h_p^e = e_h^c(p) - \frac{\Delta t}{m_p} U_p \cdot F_{pc} \]

\[ H_p^e = e_h^c(p) - \frac{\Delta t}{m_p} (U_p \cdot F_{pc} - V_c \cdot \mathcal{F}_{pc}) = e_h^c(p) - \frac{\Delta t}{m_p} \sum_{q \in Q(\partial \omega_c)} V_q \cdot \mathcal{F}_{q}^p \]
Properties

\[ \sum_{p \in Q(\partial \omega_c)} m_p^c h_p^\rho = \sum_{p \in Q(\partial \omega_c)} m_p^c H_p^\rho \]

\[ \sum_{p \in Q(\partial \omega_c)} m_p^c h_p^\mu = \sum_{p \in Q(\partial \omega_c)} m_p^c H_p^\mu \]

\[ \sum_{p \in Q(\partial \omega_c)} m_p^c h_p^e = \sum_{p \in Q(\partial \omega_c)} m_p^c H_p^e \]

Mimic the first-order scheme

1. \[ \sum_{p \in Q(\partial \omega_c)} \mathcal{F}_{pc} = 0 \]

2. \[ \sum_{q \in Q(\partial \omega_c)} \mathcal{F}_p^q = \sum_{q \in Q(\partial \omega_c)} P_h^c(p) l_q^n n_q^n - M_q c(V_q^p - U_h^c(p)) \]

3. \[ \sum_{q \in Q(\partial \omega_c)} V_q^p \cdot \mathcal{F}_q^p = P_h^c(p) \sum_{q \in Q(\partial \omega_c)} V_q^p \cdot l_q^n n_q^n - \sum_{q \in Q(\partial \omega_c)} V_q^p \cdot M_q c(V_q^p - U_h^c(p)) \]
Artificial cell velocity and subcell forces

\[ \mathbf{S}_{pc} = P_h^c(p)l_{pc}^n n_{pc}^n + (M_c - M_{qc})(\mathbf{V}_c - \mathbf{U}_h^c(p)) \]

where \( M_c = \sum_{p \in Q(\partial \omega_c)} M_{pc} \)

\[ \mathbf{V}_c = \frac{1}{N_Q - 1} M_c^{-1} \sum_{q \in Q(\partial \omega_c)} \left( (M_c - M_{qc})\mathbf{U}_h^c(q) - P_h^c(q)l_{qc}^n n_{qc}^n \right) \]

where \( N_Q = |Q(\partial \omega_c)| = N_P (d - 1) \) and \( N_P = |P(\omega_c)| \)

Convex combination

\[ W_{c+1}^{n+1} = \frac{1}{m_c} \left( m_c^* W_c^* + \sum_{p \in Q(\partial \omega_c)} m_c^p H_c^p \right), \]

where \( H_c^p = (H_p^\rho, H_p^u, H_p^e)^t \) and \( m_c = m_c^* + \sum_{p \in Q(\partial \omega_c)} m_c^p \)
Finally, for the high-order cell-centered Lagrangian schemes presented, if

1. $W^n_c \in G$, $W^c_\star \in G$ and $\forall p \in Q(\partial \omega_c)$, $W^c_h(p) \in G$

2. $\Delta t \leq C_v \frac{m^c_p \left(\frac{1}{\rho}\right)^c_h(p)}{|(U^n_p - V_c). l_{pc}^n n_{pc}^n|}$, with $C_v < \min \left(1, \frac{\varepsilon^c_h(p)}{|P^c_h(p)| \left(\frac{1}{\rho}\right)^c_h(p)}\right)$

3. $\Delta t \leq \frac{1}{2} \sum_j Z_j l_j = \frac{m^c_p}{m^c_p} \frac{v^n_c}{a_c L_c}$

Then $W^{n+1}_c \in G$
Quantities involved

- $\forall p \in Q(\partial \omega_c), \ W^c_h(p) \in G$

\[
W^c_h = \frac{\sum_{q \in R_c \backslash Q(\partial \omega_c)} m^c_q W^c_{h}(q)}{\sum_{p \in R_c \backslash Q(\partial \omega_c)} m^c_p} \in G \quad \text{or} \quad \forall q \in R_c \backslash Q(\partial \omega_c), \ W^c_{h}(q) \in G
\]

Positive limitation

- $\tilde{\rho}^c_h = (\frac{1}{\rho})_c + \theta_{\rho} ((\frac{1}{\rho})_h - (\frac{1}{\rho})_c)$
- $\tilde{U}^c_h = U^c_c + \theta_{\varepsilon} (U^c_c - U^c_c)$
- $\tilde{e}^c_h = e^c_c + \theta_{\varepsilon} (e^c_c - e^c_c)$

where $\theta_{\rho} \in [0, 1]$ and $\theta_{\varepsilon} \in [0, 1]$
### Riemann invariants differentials

- \( \frac{d\alpha}{dt} = \frac{dU}{t} \)
- \( \frac{d\alpha}{d\nu} = \frac{d\left(\frac{1}{\rho}\right)}{d\nu} - \frac{1}{\rho a} \frac{dU}{n} \)
- \( \frac{d\alpha}{d\rho} = \frac{d\left(\frac{1}{\rho}\right)}{d\rho} + \frac{1}{\rho a} \frac{dU}{n} \)
- \( \frac{d\alpha}{de} = \frac{de - U}{dU} + P \frac{d\left(\frac{1}{\rho}\right)}{d\rho} \)

### Mean value linearization

- \( \alpha^{c}_{t,h} = U^{c}_{t} \cdot t \)
- \( \alpha^{-,h} = \left(\frac{1}{\rho}\right)_{h}^{c} - \frac{1}{Z_{c}} U^{c}_{h} \cdot n \)
- \( \alpha^{c}_{+,h} = \left(\frac{1}{\rho}\right)_{h}^{c} + \frac{1}{Z_{c}} U^{c}_{h} \cdot n \)
- \( \alpha^{c}_{e,h} = e^{c}_{h} - U^{c}_{0} \cdot U^{c}_{h} + P^{c}_{0} \left(\frac{1}{\rho}\right)_{h}^{c} \)

### Unit direction ensuring symmetry preservation

- \( n = \frac{U^{c}_{0}}{\|U^{c}_{0}\|} \) and \( t = e_{z} \times \frac{U^{c}_{0}}{\|U^{c}_{0}\|} \)

### Double specific volume limitation

- Standard limitation on \( \left(\frac{1}{\rho}\right)_{h} \) and on the Riemann invariants are performed
- Only the most limiting procedure is retained to avoid spurious oscillations
Stability

- Same stability results on the piecewise constant part $W_c$ of the numerical solution $W_h^c$ as for the first-order schemes
- To obtain the same stability properties on the whole piecewise polynomial solution $W_h$, the limitation at time $t^n$ has to ensure that

$$\forall x \in \omega, \quad W_h(x) \in G$$

Then

- $\| (\frac{1}{\rho})_h^n \|_{L_1} = \| \frac{1}{\rho_0} \|_{L_1}$
- $\| e_h^n \|_{L_1} = \| e^n_0 \|_{L_1}$
- $\| K_h^n \|_{L_1} < \| e_h^n \|_{L_1}$
- $\| \mathbf{U}_h^n \|_{L_2}^2 < m_\omega + \| e_h^n \|_{L_2}^2$
1. Cell-Centered Lagrangian schemes

2. Lagrangian and Eulerian descriptions

3. Compatible first-order positivity-preserving discretization

4. High-order positivity-preserving extension

5. CCDG numerical results

6. Conclusion
Cylindrical Sod shock problem

(a) Initial time $t = 0$

(b) Final time $t = 1$

Figure: Density maps on a 100x5 polar mesh, with the second-order DG scheme
Cylindrical Sod shock problem

Figure: Density profile on a 100x5 polar mesh, at final time $t = 1$
Sedov point blast problem on a Cartesian grid

(a) Initial time $t = 0$

(b) Final time $t = 1$

Figure: Pressure maps on a 30x30 Cartesian mesh, with the second-order DG scheme
Sedov point blast problem on a Cartesian grid

(a) Density profiles

(b) Pressure profiles

Figure: Density and pressure profiles on a 30x30 Cartesian mesh, at final time $t = 1$
Sedov point blast problem on a polygonal grid

**Figure:** Final grids on mesh made of 775 polygonal cells, with the second-order DG scheme
Sedov point blast problem on a polygonal grid

(a) Density profiles

(b) Pressure profiles

Figure: Density and pressure profiles on mesh made of 775 polygonal cells, at final time $t = 1$
Cylindrical Sedov point blast problem

(a) Initial time $t = 0$

(b) Final time $t = 1$

**Figure:** Final grids on a 30x5 polar mesh, with the second-order DG scheme
Cylindrical Sedov point blast problem

(a) Density profiles

(b) Pressure profiles

Figure: Density and pressure profiles on a 30x5 polar mesh, at final time $t = 1$
Noh problem

(a) 1st order

(b) 2nd order

Figure: Final grids on a Cartesian grid made of 50 $\times$ 50 cells, at final time $t = 0.6$
Noh problem

Figure: Density profile on a Cartesian grid made of $50 \times 50$ cells, at final time $t = 0.6$
Cylindrical Noh problem

(a) 1st order

(b) 2nd order

Figure: Final grids on a 50x5 polar mesh, at final time $t = 0.6$
Cylindrical Noh problem

Figure: Density profile on a 50x5 polar mesh, at final time $t = 0.6$
Saltzman problem

Figure: Final grids on a 10x100 deformed Cartesian mesh, at time $t = 0.6$
Saltzman problem

Figure: Density and pressure profiles on a 10x100 deformed Cartesian mesh, at time $t = 0.6$
Saltzman problem

Figure: Final grids on a 10x100 deformed Cartesian mesh, at time $t = 0.9$
Saltzman problem

Figure: Density and pressure profiles on a 10x100 deformed Cartesian mesh, at time $t = 0.9$
Taylor-Green vortex problem

(a) 1st order

(b) 2nd order

Figure: Final grids at final time $t = 0.75$, on a 10x10 Cartesian mesh
## Taylor-Green vortex problem

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<th>$E^h_{L_2}$</th>
<th>$q^h_{L_2}$</th>
<th>$E^h_{L_\infty}$</th>
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**Table**: Rate of convergence computed on the velocity at time $t = 0.1$
Taylor-Green vortex problem

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**Table:** Rate of convergence computed on the velocity at time $t = 0.1$
Air-water-air problem

(a) Initial time \( t = 0 \)

(b) Final time \( t = 7 \times 10^{-3} \)

Figure: Density maps on a 120x9 polar mesh, for the second-order DG scheme
Air-water-air problem

Figure: Pressure maps on a 120x9 polar mesh, for the second-order DG scheme

(a) Initial time $t = 0$

(b) Final time $t = 7E-3$
Air-water-air problem

(a) Density profile

(b) Normal velocity profile

Figure: Density and normal velocity profiles on a 120x9 polar mesh, at final time $t = 7E-3$
Underwater TNT charge explosion

(a) Initial time $t = 0$
(b) Final time $t = 2.5E-4$

Figure: Density maps on a 120x9 polar mesh, for the second-order DG scheme
Underwater TNT charge exlosion

(a) Density profile
(b) Pressure profile

Figure: Density and pressure profiles on a 120x9 polar mesh, at final time $t = 2.5E-4$
1. Cell-Centered Lagrangian schemes

2. Lagrangian and Eulerian descriptions

3. Compatible first-order positivity-preserving discretization

4. High-order positivity-preserving extension

5. CCDG numerical results

6. Conclusion
Conclusions

- Demonstration of the positivity-preserving criteria of a whole class of cell-centered Lagrangian scheme, under particular time step constraints, for different EOS (such as ideal gas, stiffened-water or detonation JWL)
- Extension of the demonstration to high-order of accuracy, under particular limitation of the solution
- Demonstration of $L_1$ stability of the specific volume and total energy
- Control of the $L_1$ norm of the kinetic energy and of the $L_2$ norm of the velocity
- Improvement of the robustness

Perspectives

- Extension of the numerical applications to higher-order of accuracy
- Extension of the CCDG to solid dynamics such as hyperelasticity
